Covariant (Invariant) form of Lorentz force

Lorentz force density or Lorentz force acting on unit volume of any charge distribution of charge density ρ in electromagnetic field, is given by

$$\vec{f} = \rho \vec{E} + \vec{J} \times \vec{B} \quad \text{——— (1)}$$

where $\vec{E} = E_1 \hat{i} + E_2 \hat{j} + E_3 \hat{k}$ = Electric field

$\vec{B} = B_1 \hat{i} + B_2 \hat{j} + B_3 \hat{k}$ = Magnetic field

and $\vec{J} = J_1 \hat{i} + J_2 \hat{j} + J_3 \hat{k}$ = current density

Equⁿ (1) can be written in component form as

$$f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k} = \rho (E_1 \hat{i} + E_2 \hat{j} + E_3 \hat{k}) + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ J_1 & J_2 & J_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$\Rightarrow f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k} = \rho E_1 \hat{i} + \rho E_2 \hat{j} + \rho E_3 \hat{k} + \hat{i}(J_2 B_3 - J_3 B_2) - \hat{j}(J_1 B_3 - J_3 B_1) + \hat{k}(J_1 B_2 - J_2 B_1) \quad \text{——— (2)}$$

Equating coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$$f_1 = \rho E_1 + J_2 B_3 - J_3 B_2 \quad \text{——— (3)}$$

$$f_2 = \rho E_2 + J_3 B_1 - J_1 B_3 \quad \text{——— (4)}$$

$$f_3 = \rho E_3 + J_1 B_2 - J_2 B_1 \quad \text{——— (5)}$$

Electromagnetic field tensor $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix} = \begin{bmatrix} 0 & B_3 & -B_2 & -\frac{i}{c}E_1 \\ -B_3 & 0 & B_1 & -\frac{i}{c}E_2 \\ B_2 & -B_1 & 0 & -\frac{i}{c}E_3 \\ \frac{i}{c}E_1 & \frac{i}{c}E_2 & \frac{i}{c}E_3 & 0 \end{bmatrix}$$

Equⁿˢ (3) to (5) can be written in terms of electromagnetic field tensor $F_{\mu\nu}$ as

$$f_1 = F_{11} J_1 + F_{12} J_2 + F_{13} J_3 + F_{14} J_4 \quad \text{——— (6)}$$

$$f_2 = F_{21} J_1 + F_{22} J_2 + F_{23} J_3 + F_{24} J_4 \quad \text{——— (7)}$$

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$$f_2 = F_{21}J_1 + F_{22}J_2 + F_{23}J_3 + F_{24}J_4 \text{ —— } (A_2)$$

where $J_4 = ic\rho$ and current density four vector J_ν is

$$J_\nu = (\vec{J}, ic\rho)$$

Eqns (A_1) to (A_3) can be written more compactly in the form of single equation as

$$f_\alpha = F_{\alpha\nu} J_\nu \quad \text{where } \alpha = 1,2,3, \text{ —— } (A)$$

$\because$ Here $F_{\alpha\nu} J_\nu$ is space component of a four vector so f_α must be space part of a four vector

$$\Rightarrow f_\alpha = f_\mu$$

So eqn (A) will become

$$f_\mu = F_{\mu\nu} J_\nu \text{ —————— } (B)$$

f_μ is known as force-density four vector.

Eqn (B) may be written as

$$f_\mu = \frac{1}{\mu_0} F_{\mu\nu} \left(\frac{\partial F_{\nu\sigma}}{\partial x_\sigma} \right) \text{ —————— } (C)$$

$$\because \frac{\partial F_{\nu\sigma}}{\partial x_\sigma} = \mu_0 J_\nu$$

Here eqns (B) and (C) are not different because eqn (C) is written by replacing J_ν by $\frac{1}{\mu_0}\left(\frac{\partial F_{\nu\sigma}}{\partial x_\sigma}\right)$. So eqns (B) and (C) are identical.

Eqn (C) is a tensor equation of rank 1.

Since a tensor equation is always invariant (covariant) under Lorentz transformation, so equations defined in (B) and (C) must be invariant under Lorentz transformation.

Thus eqn (B) or (C) is covariant (invariant) form of Lorentz force equation.

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Equ'ⁿˢ (B) or (C) refers the rate of change in mechanical momentum per unit volume as its space part and mechanical energy per unit volume as its time part.

* physical meaning of the fourth component (f_4) of the force density four vector (f_μ) :-

Force-density four vector is defined as

$$f_\mu = F_{\mu\nu} \cdot J_\nu$$

So fourth component (f_4) of force-density four vector f_μ will be

$$f_4 = F_{4\nu} \cdot J_\nu \quad \text{where } \nu = 1, 2, 3, 4.$$

$$\Rightarrow f_4 = F_{41} \cdot J_1 + F_{42} \cdot J_2 + F_{43} \cdot J_3 + F_{44} J_4$$

$$\Rightarrow f_4 = \frac{i}{c} E_1 J_1 + \frac{i}{c} E_2 J_2 + \frac{i}{c} E_3 J_3 + 0 \cdot J_4$$

$$\left(\because F_{41} = \frac{i}{c} E_1 \,,\; F_{42} = \frac{i}{c} E_2 \,,\; F_{43} = \frac{i}{c} E_3 \text{ and } F_{44} = 0\right)$$

$$\Rightarrow f_4 = \frac{i}{c} (E_1 J_1 + E_2 J_2 + E_3 J_3)$$

$$\Rightarrow \boxed{f_4 = \frac{i}{c} (\vec{E} \cdot \vec{J})} \quad \text{——— (D)}$$

It is fourth component of force-density four vector.

Therefore, the fourth component of force-density four vector is imaginary.

Equ'ⁿ (D) represents the amount of work done by the electric field on the charge per unit volume per unit charge, with a factor of $\frac{i}{c}$.

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